DIFFERENTIAL EQUATIONS

Method Part(a) e.g. 10(a) 1993

$$\left(x^2+2\right)\frac{dy}{dx}=x\left(y+1\right)$$

(1) Separate variables keeping x's on one side and y's on the other

$$\frac{dy}{y+1}=\frac{xdx}{x^2+2}$$

*(2) Integrate both sides (**you must know all integrated maths types from honours maths course to answer this – also tables p.g. 41)*

$$\int\frac{dy}{y+1}=\int\frac{xdx}{x^2+2}$$

$$\int\frac{dy}{y+1}=\frac{1}{2}\int\frac{du}{u}$$

$$\ln\left|y+1\right|=\frac{1}{2}\ln|u|+c$$

$$\ln\left|y+1\right|=\frac{1}{2}\ln\left|x^2+2\right|+c$$

Let:

$$u=x^2+2$$

$$\frac{du}{dx}=2x$$

$$\frac{du}{2}=xdx$$

(3) Use initial conditions to find c (- if there are no initial conditions given you must have +c in your answer)

$$\ln\left|y+1\right|=\frac{1}{2}\ln\left|x^2+2\right|+c$$

but y=2 when x=1

$$\Rightarrow\ln 3=\frac{1}{2}\ln 3+c$$

$$\ln 3=\ln 3^{1/2}+c$$

$$c=\ln 3-\ln 3^{1/2}$$

$$c=\frac{\ln 3^1}{\ln 3^{1/2}}=\ln 3^{1/2}$$

$$\boxed{\Rightarrow\ln\left|y+1\right|=\frac{1}{2}\ln\left|x^2+2\right|+\ln 3^{1/2}}$$

Method Part(b)

All of the steps outlined in part (a) also apply in part (b).

However, note the following additions:

(1) *As part (b) is on applications of differential equations you will often have to form your own differential equation from the information given.*
e.g. if you are told 'a car has a retardation which is proportional to velocity squared' then the equation is

$$a = -kv^2 \text{ or } \frac{dv}{dt} = -kv^2$$

(2) *You will usually have to find the initial conditions from the information given also.*
e.g. 'a car starts from rest' means: $v = 0$ *at* $t = 0$

(3) *From the chain rule:* $a = \frac{dv}{dt} = \frac{ds}{dt}\frac{dv}{ds} = v\frac{dv}{ds}$

In some questions you will need to use $a = \frac{dv}{dt}$ *and in others you use:* $a = v\frac{dv}{ds}$.

<u>*How do you know which to use?*</u>
The question really gives a clue e.g. 'find velocity when displacement is 10m' – since velocity (v) and displacement (s) are mentioned then using $a = v\frac{dv}{ds}$ *is probably the quickest way forward. On the other hand if the question says 'find velocity when t=5secs' then you should try* $a = \frac{dv}{dt}$ *first. Often you will have to integrate twice.*
e.g. using $\frac{dv}{dt}$ *will give an equation for v in terms of t. Then by letting* $v = \frac{ds}{dt}$ *you can integrate again to get s in terms of t.*
Likewise using $a = v\frac{dv}{ds}$ *gives an expression for v in terms of s and by letting* $v = \frac{ds}{dt}$ *again you get an expression for s in terms of t. In summary if you pick the right equation at the start it will lead to the quickest solution but if you pick the wrong one it will usually lead to the same solution except it will be less straightforward. On rare occasions if you use the wrong equation you may get an expression which you can't integrate using any of the standard methods – if this happens in the exam <u>Don't Panic!</u> – just go back to the start and try the other equation.*

(4) *The following results are useful in part(b)*

(i) $F = ma \Rightarrow F = m(\frac{dv}{dt}) = mv\frac{dv}{ds}$
For a resistance force F is negative.

(ii) When a particle is projected vertically upwards the air resistance is proportional to the velocity of the particle i.e. $F = -kv^n$ *(Stokes Law)*
$\Rightarrow$ *Total Force including gravity* $= -mg - kv^n$
However when it's falling downwards the total force is $mg - kv^n$ *(air resistance always opposes motion)*

In Summary

Upwards: Net force equation
$F = ma = -mg - kv^n$
Downwards: Net force equation
$F = ma = mg - kv^n$

(iii) Power $P = \frac{w}{t}$ $\left(\frac{work}{time}\right)$

But Work = Fxd (force x dist)

$\Rightarrow P = \frac{F.d}{t} = F.\frac{d}{t}$ $\left(\frac{dist}{time}\right)$

P = F.v (force x velocity)

In general when a body is travelling under a tractive force T at a speed v, the power developed = Tv watts

(5) *Know rules of logs and indices!*

<u>Note:</u>
$\ln x = y \quad \Rightarrow e^y = x$
$\Rightarrow e^{\ln x} = x$ i.e. $e^{\ln}$ always cancels.

e.g. $\ln\left|g - kv^2\right| = \ln g - 2kh \rightarrow$ (*find V*)

$$\Rightarrow \quad e^{(\ln g - 2kh)} = g - kv^2$$

$$\frac{e^{\ln g}}{e^{2kh}} = g - kv^2 \qquad \text{(since } \frac{a^m}{a^n} = a^{m-n}\text{)}$$

$$\frac{g}{e^{2kh}} = g - kv^2 \qquad (e^h \text{ cancels})$$

$$kv^2 = g - g/e^{2kh}$$

$$v^2 = g(1 - 1/e^{2kh})$$

$$v = \sqrt{g(1 - 1/e^{2kh})}$$